Mathematics
Class XII
Sample Paper – 8

Time: 3 hours **Total Marks: 100**

1. All questions are compulsory.
2. The question paper consist of 29 questions divided into three sections A, B, C and D. Section A comprises of 4 questions of one mark each, section B comprises of 8 questions of two marks each, section C comprises of 11 questions of four marks each and section D comprises of 6 questions of six marks each.
3. Use of calculators is not permitted.

SECTION – A

1. Write the position of the element 6 in the given matrix, and denote it as a_{ij}.

$$\begin{bmatrix} 1 & 16 & 8 & 9 \\ 7 & 5 & 3 & 2 \\ 4 & 10 & 6 & 11 \end{bmatrix}$$

2. Find $\frac{dy}{dx}$, if 2x + 3y = cos x

3. Is the differential equation given by $s^2\frac{d^2y}{dx^2}+sy\frac{dy}{dx}=s$, linear or nonlinear. Give reason.

4. The Cartesian equations of a line are $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Find a vector equation for the line.

OR

Find the angle between following pairs of line

$\frac{x+4}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$

SECTION – B

5. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by a * b = min {a, b}. Write the operation table of the operation *.

6. Solve the matrix equation

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

7. Evaluate:

$$\int \frac{5x-2}{1+2x+3x^2}dx$$

8. Evaluate:

$$\int \frac{x^2}{x^2+4 \quad x^2+9}dx$$

OR

Evaluate:

$$\int \frac{(x+3)e^x}{(x+5)^3}dx$$

9. Form differential equation of the family of curves y = a sin (bx + c), a and c being parameters.

10. ABCD is a parallelogram with $\overrightarrow{AB}=2\hat{i}-4\hat{j}+5\hat{k}; \overrightarrow{AD}=\hat{i}-2\hat{j}-3\hat{k}$

Find a unit vector parallel to its diagonal $\overrightarrow{AC}$. Also, find the area of the parallelogram ABCD

OR

Find the projection of $(\vec{b}+\vec{c})$ on $\vec{a}$, where $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$.

11. Four cards are drawn successively with replacement from a well shuffled deck of 52 cards. What is the probability that

(i) All the four cards are spades?

(ii) Only 3 cards are spades?

(iii) None is a spade?

12. A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Determine: (i) k (ii) $P(X < 3)$ (iii) $P(X > 6)$ (iv) $P(1 \leq X < 3)$

OR

A and B throw a dice alternatively till one of them gets a ‘6’ and wins the game. Find their respective probabilities of winning, if A starts first.

SECTION – C

13. Let A = Q x Q and let * be a binary operation on A defined by (a, b)*(c, d) = (ac, b + ad) for (a, b), (c, d) $\in$ A. Determine whether * is Commutative and associative. Then, with respect to * on A

(i) Find the identify element in A.

(ii) Find the invertible elements of A.

OR

Let A = Q × Q, Q being the set of rational. Let '*' be a binary operation on A, defined by (a, b) * (c, d) = (ac, ad + b). Show that

(i) '*' is not commutative

(ii) '*' is associative

(iii) The identity element with respect to '*' is (1, 0)

14. Write in the simplest form:

$$y=\cot^{-1}\left(\sqrt{1+x^2}-x\right)$$

15. Let a, b, and c be positive numbers , but not equal and not all are zero.

Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative

16. If sin y = x sin (a + y), prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$

OR

If $y=b\tan^{-1}\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right)$, find $\frac{dy}{dx}$

17. If $y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$, then show that $\frac{dy}{dx}=\sqrt{\frac{3}{\cos x\cos 3x}}$

18. A man of height 2 m walks at a uniform speed of 5 km/h away from a lamp post which is 6 m high. Find the rate at which the length of his shadow increases.

19. Evaluate:

$$\int\frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx$$

20. Evaluate: $\int_{-1}^{2}(7x-5)dx$, as a limit of sums.

21. Solve the initial value problem: cos (x + y)dy = dx, y(0) = 0.

OR

Solve: $(x+y)^2\frac{dy}{dx}=a^2$

22.

a) If $\hat{i}.\hat{j}.\hat{k}$ represents the right handed system of mutually perpendicular vectors and $\vec{\alpha}=3\hat{i}-\hat{j}$; $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{\beta}$ in the form of $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$ where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

b) Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors of magnitude 3, 4 and 5 units respectively. If each of these is perpendicular to the sum of the other two vectors, find $\left|\vec{a}+\vec{b}+\vec{c}\right|$.

23.

Find the coordinates of the point, where the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ intersects the plane $x - y + z - 5 = 0$. Also find the angle between the line and the plane.

SECTION – D

24. If $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 2\\ 2 & 0 & 3\end{bmatrix}$, then show that A is a root of polynomials $f(x) = x^3 - 6x^2 + 7x + 2$

OR

If $A=\begin{bmatrix}-1 & -1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}$, show that $AB \neq BA$

25. Find the volume of the largest cylinder which can be inscribed in a sphere of radius r.

26. Find the area of the region

$\{(x, y): 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

OR

Calculate the area

(i) between the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the x-axis between x = 0 to x = a

(ii) Triangle AOB is in the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

Where OA = a and OB = b.

Find the area enclosed between the chord AB and the arc AB of the ellipse

(iii) Find the ratio of the two areas found.

27. Find the Cartesian equation of the plane passing through the points A(0, 0, 0) and B(3, -1, 2) and parallel to the line $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$

OR

Find the equation of the line passing through the point (-1,3,-2) and perpendicular to the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$.

28. A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs. 8000 on each piece of model A and Rs. 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realize a maximum profit? What is the maximum profit per week?

29. Two bags A and B contain 3 red and 4 black balls, and 4 red and 5 black balls respectively. From bag A, one ball is transferred to bag B and then a ball is drawn from bag B. The ball is found to be red in colour. Find the probability that

(a)The transferred ball is black ?

(b) The transferred ball is red ?

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SECTION – A

1. The element '6' lies on 3rd row and 3rd column
So,
$a_{33} = 6$

2. 2x + 3y = cos x
Differentiating w.r.t. x, we get,

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}\cos x$$

$$2+3\frac{dy}{dx}=-\sin x$$

$$\frac{dy}{dx}=\frac{-\sin x-2}{3}$$

3. DE:

$$s^2\frac{d^2y}{dx^2}+sy\frac{dy}{dx}=s$$

It is nonlinear, since we have product of dependent variable and differential coefficient $y\frac{dy}{dx}$

4. $$\frac{x-5}{3}=\frac{y-(-4)}{7}=\frac{z-6}{2}$$

Clearly, it passes through (5, -4, 6) and has a direction ratios proportional to 3, 7, 2.
So its vector equation is

$$\vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

OR

Let θ be the angles between, the given two lines
So, the angle between them given their direction cosines is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}\sqrt{{a_2}^2+{b_2}^2+{c_2}^2}}$$

substituting we get

$$\theta=\cos^{1}\left(\frac{8}{5\sqrt{3}}\right)$$

SECTION – B

5. The binary operation * on the set {1, 2, 3, 4, 5} is defined by a * b = min {a, b}
The operation table for the given operation * on the given set is as follows

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

6. We have,

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x^2 - 3x \\ y^2 - 6y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$x^2 - 3x = -2$$

$$y^2 - 6y = 9$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 2 \text{ or } x = 1$$

$$y^2 - 6y - 9 = 0$$

$$y = \frac{6 \pm \sqrt{36+36}}{2} = 3 \pm 3\sqrt{2}$$

7. $\int \frac{5x-2}{1+2x+3x^2}dx$

$$=5\int \frac{x-\frac{2}{5}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x-\frac{12}{5}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x+2-\frac{12}{5}-2}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x+2-\frac{22}{5}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x+2}{1+2x+3x^2}dx-\frac{5}{6}\times\frac{22}{5}\int \frac{1}{3\left\{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right\}}dx$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}dx$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\times\frac{3}{\sqrt{2}}\tan^{-1}\left[\frac{\left(x+\frac{1}{3}\right)}{\frac{\sqrt{2}}{3}}\right]+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\times\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

8. Let $x^2 = y$

$$\frac{x^2}{x^2+4\ \ x^2+9} = \frac{y}{y+4\ \ y+9} = \frac{A}{y+4} + \frac{B}{y+9}$$

$y = A(y+9) + B(y+4)$

Comparing both sides,

$A + B = 1$ and $9A + 4B = 0$

Solving, we get $A = \frac{-4}{5}$ and $B = \frac{9}{5}$

$$\therefore I = \int \left[\frac{-4}{5\ x^2+4} + \frac{9}{5\ x^2+9} \right] dx$$

$$= -\frac{4}{5} \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5} \times \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= -\frac{2}{5} \tan^{-1}\frac{x}{2} + \frac{3}{5} \tan^{-1}\frac{x}{3} + C$$

OR

$$\int \frac{(x+3)e^x}{(x+5)^3} dx$$

$$= \int \frac{(x+5-2)e^x}{(x+5)^3} dx$$

$$= \int \left[\frac{(x+5)}{(x+5)^3} - \frac{2}{(x+5)^3} \right] e^x.dx$$

$$= \int \left[\frac{1}{(x+5)^2} - \frac{2}{(x+5)^3} \right] e^x.dx$$

This is of the form

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

$$\Rightarrow \int \left[\frac{1}{(x+5)^2} - \frac{2}{(x+5)^3} \right] e^x.dx$$

$$= \frac{e^x}{(x+5)^2} + C$$

9. We have to differentiate it w.r.t. x two times

differentiating

$$\frac{dy}{dx} = ab\cos(bx+c) \ldots\ldots.(1)$$

differentiating again

$$\frac{d^2y}{dx^2} = -ab^2\sin(bx+c) \ldots\ldots\ldots(2)$$

$$\frac{d^2y}{dx^2} = -b^2y \ldots\ldots\ldots\ldots\{\because y = a\sin(bx+c)\}$$

which is the required differential equation

10. ABCD is a parallelogram with,

$$\overrightarrow{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}; \overrightarrow{AD} = \hat{i} - 2\hat{j} - 3\hat{k}$$

Using the parallelogram law of vector addition, diagonal is given by

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \ 2\hat{i} - 4\hat{j} + 5\hat{k} \ + \ \hat{i} - 2\hat{j} - 3\hat{k} \ = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Unit vector parallel to diagonal $\overrightarrow{AC}$

$$= \frac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{|3\hat{i} - 6\hat{j} + 2\hat{k}|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + (2)^2}}$$

$$= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{49}} = \frac{1}{7} \ 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Area of the parallelogram ABCD= $|\overrightarrow{AB} \times \overrightarrow{AD}|$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= |\hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)| = |\hat{i}(22) - \hat{j}(-11) + \hat{k}(0)| = |\hat{i}(22) + \hat{j}(11)|$$

$$= \sqrt{(22)^2 + (11)^2 + 0^2} = 11\sqrt{5} \text{ sq units}$$

OR

We have $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Now, $\vec{b} + \vec{c} = \hat{i} + 2\hat{j} - 2\hat{k} + 2\hat{i} - \hat{j} + 4\hat{k}$

$= 3\hat{i} + \hat{j} + 2\hat{k}$

$|\vec{a}| = |2\hat{i} - 2\hat{j} + \hat{k}| = \sqrt{4+4+1} = 3$

Also, $\vec{a} \cdot (\vec{b} + \vec{c}) = (2\hat{i} - 2\hat{j} + \hat{k}) \cdot (3\hat{i} + \hat{j} + 2\hat{k})$

$= 6 - 2 + 2 = 6$

So projection of $(\vec{b} + \vec{c})$ on vector $\vec{a}$ = $\dfrac{\vec{a}.(\vec{b} + \vec{c})}{|\vec{a}|} = \dfrac{6}{3} = 2$ units

11. This is a case of Bernoulli trials.

p = P(Success) = P(getting a spade in a single draw) $= \dfrac{13}{52} = \dfrac{1}{4}$

$q = P(\text{Failure}) = 1 - p = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

(i)All the four cards are spades = P(X = 4) = ${}^4C_4 p^4 q^0 = \left(\dfrac{1}{4}\right)^4 = \dfrac{1}{256}$

(ii)Only 3 cards are spades= P(X = 3) = ${}^4C_3 p^3 q^1 = \dfrac{12}{256} = \dfrac{3}{64}$

(iii)None is a spade = P(X = 0) = ${}^4C_0 p^0 q^4 = \left(\dfrac{3}{4}\right)^4 = \dfrac{81}{256}$

12. (i) $\sum_{i=0}^{7} P(X_i)=1$

$\Rightarrow \left[0+k+2k+2k+3k+k^2+2k^2+7k^2+k\right]=1$

$\Rightarrow 10k^2+9k=1 \Rightarrow 10k^2+9k-1=0 \Rightarrow 10k^2+10k-k-1=0$

$\Rightarrow 10k(k+1)-(k+1)=0$

$\Rightarrow (k+1)(10k-1)=0$

$\Rightarrow k=-1, k=\frac{1}{10}$

k, also being a probability cannot be negative

$\Rightarrow k=\frac{1}{10}$

(ii) $P(X<3)=P(0)+P(1)+P(2)=0+k+2k=3k=\frac{3}{10}$

(iii) $P(X>6)=P(7)=7k^2+k=7\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)=\frac{17}{100}$

(iv) $P(1\leq X<3)=P(1)+P(2)=k+2k=3k=\frac{3}{10}$

OR

Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus, $P(S) = \frac{1}{6}$; $P(F) = \frac{5}{6}$

P(A wins in I throw) = P(S) = $\frac{1}{6}$

P(A wins in III throw) = P(FFS) = $\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

P(A wins in v throw) = P(FFFFS) = $\left(\frac{5}{6}\right)^4\times\frac{1}{6}$

P(A wins) = P(S) + P(FFS) + P(FFFFS) = $\frac{1}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\left(\frac{5}{6}\right)^4\times\frac{1}{6}+\ldots$

$=\frac{\frac{1}{6}}{1-\frac{25}{36}}=\frac{6}{11}$

P(B wins) = $1-\frac{6}{11}=\frac{5}{11}$

SECTION – C

13. Given that A = Q × Q and (a, b) * (c, d) = (ac, b + ad)

(i) We know that $a*e=a$ for identity element.

Let, $a=(a_1,a_2)$ and $e=(e_1,e_2)$

$\Rightarrow a*e=(a_1e_1,a_1e_2+a_2)$

It should be equals (a_1,a_2)

$(a_1e_1,a_1e_2+a_2)=(a_1,a_2)$ at $e=(1,0)$

Hence, $e=(1,0)$ satisfies condition.

(ii) Condition for invertible element $a*b=b*a=e$

Let, $a(a_1,a_2)$ and $b(x_1,x_2)$

$a*b=(a_1x_1,a_1x_2+a_2)=(1,0)$

This will satisfy when $x_1=\dfrac{1}{a_1}$ and $x_2=\dfrac{-a_2}{a_1}$

Hence, invertible element $=\left(\dfrac{1}{a_1},\dfrac{-a_2}{a_1}\right)$

OR

(a, b) * (c , d) = (ac, ad + b)

(c, d) * (a, b) = (ca, cb + d)

(ac, ad + b) $\neq$ (ca, cb + d)

So, '*' is not commutative

Let (a, b) (c, d), (e, f) $\in$ A, Then

((a, b)* (c, d)) * (e, f) = (ac , ad + b) * (e, f) = ((ac) e , (ac) f + (ad + b))

= (ace , acf + ad + b)

(a, b)* ((c, d)* (e, f)) = (a, b) * (ce , cf + d) =

(a (ce) , a (cf + d) +b) = (ace , acf + ad + b)

((a, b)* (c, d)) * (e, f) = (a, b) * ((c, d)* (e, f))

Hence, '*' is associative.

Let (x, y) $\in$A. Then (x, y) is an identity element, if and only if

(x, y) * (a, b) = (a, b) = (a, b) * (x, y), for every (a, b) $\in$A

Consider (x, y) * (a, b) = (xa, xb + y)

(a, b) * (x, y)= (ax, ay +b)

(xa , xb + y) = (a, b) = (ax, ay +b)

ax = x a = a $\Rightarrow$ x = 1

xb + y = b = ay +b $\Rightarrow$ b + y = b = ay + b $\Rightarrow$ y = 0 = ay $\Rightarrow$ y = 0

Therefore, (1, 0) is the identity element

14.

Let $y = \cot^{-1}\left(\sqrt{1+x^2} - x\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \cot^{-1}\left(\sqrt{1+\tan^2\theta} - \tan\theta\right)$$

$$y = \cot^{-1}\left(\sec\theta - \tan\theta\right)$$

$$y = \cot^{-1}\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)$$

$$y = \cot^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)$$

$$y = \cot^{-1}\left[\frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)}\right]$$

$$y = \cot^{-1}\left[\frac{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]$$

$$y = \cot^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$y = \cot^{-1}\left[\cot\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$y = \frac{\pi}{4}+\frac{\theta}{2}$$

$$\therefore y = \frac{\pi}{4}+\frac{1}{2}\tan^{-1}x.$$

15. Let a, b, c be positive numbers not all are zero.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= -(a+b+c)\left[a^2 + b^2 + c^2 - ab - bc - ca\right]$$

$$= -\frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$$

$$= -\frac{1}{2}(\text{A positive real number})[\text{At least one non zero positive real number}]$$

$$= -\frac{1}{2} \times \text{positive real number}$$

$=$ A negative real number

16. Differentiating both sides of the given relation with respect to x, we get

$$\frac{d}{dx}(\sin x)=\frac{d}{dx}\{x\sin(a+y)\}$$

$$\cos y\frac{dy}{dx}=1\times\sin(a+y)+x\cos(a+y)\frac{d}{dx}(a+y)$$

$$\cos y\frac{dy}{dx}=\sin(a+y)+x\cos(a+y)\frac{dy}{dx}$$

$$\{\cos y-x\cos(a+y)\}\frac{dy}{dx}=\sin(a+y)$$

$$\frac{dy}{dx}=\frac{\sin(a+y)}{\cos y-x\cos(a+y)}$$

$$\text{put } x=\frac{\sin y}{\sin(a+y)}$$

$$\frac{dy}{dx}=\frac{\sin(a+y)}{\cos y-\frac{\sin y}{\sin(a+y)}\times\cos(a+y)}$$

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin(a+y)\cos y-\sin y\times\cos(a+y)}$$

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin(a+y-y)}$$

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin(a)}$$

OR

we have

$$y = b\tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\frac{y}{b} = \tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\tan\frac{y}{b} = \frac{x}{a} + \tan^{-1}\frac{y}{x}$$

differntiating w.r.t. x

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{x\frac{dy}{dx} - y}{x^2}$$

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a} + \frac{x\frac{dy}{dx} - y}{x^2 + y^2}$$

$$\frac{d}{dx}\left\{\frac{1}{b}\sec^2\left(\frac{y}{b}\right) - \frac{x}{x^2+y^2}\right\} = \frac{1}{a} - \frac{y}{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{a} - \frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\left(\frac{y}{b}\right) - \frac{x}{x^2+y^2}}$$

17. We have

$$y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\cos y=\sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\cos y=\sqrt{\frac{4\cos^3 x-3\cos x}{\cos^3 x}}$$

$$\cos y=\sqrt{4-3\sec^2 x}$$

$$\cos^2 y=4-3\sec^2 x$$

$$\cos^2 y=4-3\left(1+\tan^2 x\right)$$

$$\cos^2 y=4-3-3\tan^2 x$$

$$1-\cos^2 y=3\tan^2 x$$

$$\sin^2 y=3\tan^2 x$$

$$\sin y=\sqrt{3}\tan x$$

differentiating w.r.t. x

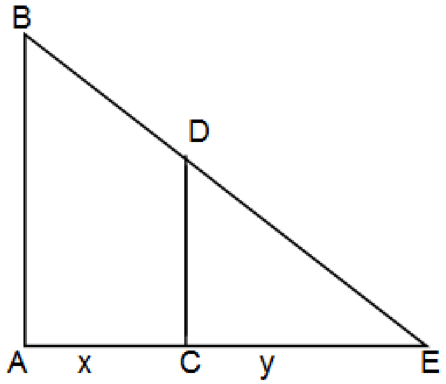
$$\cos y\frac{dy}{dx}=\sqrt{3}\sec^2 x$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{3}}{\cos y\ \cos^2 x}$$

sub, $\cos y=\sqrt{\frac{\cos 3x}{\cos^3 x}}$

$$\Rightarrow \frac{dy}{dx}=\sqrt{\frac{3}{\cos x\ \cos 3x}}$$

18.



Let AB – lamp post

CD – man

AC = x

CE = y

Given $\frac{dx}{dt} = 5 km/h$

To find $\frac{dy}{dt}$

We have $\Delta ABE \sim \Delta CDE$ (by AA)

$\Rightarrow \frac{AB}{CD} = \frac{AE}{CE}$

$\Rightarrow \frac{6}{2} = \frac{x+y}{y}$

$\Rightarrow x = 2y$

Differentiating $\frac{dx}{dt} = 2\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{5}{2} km/h$

19. $\int \frac{dx}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}$

Let $I = \int \frac{dx}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}$

Multiply the numerator and the denominator by $\sec^4 x$, we have

$$I = \int \frac{\sec^4 x dx}{\tan^4 x + \tan^2 x + 1}$$

$$I = \int \frac{\sec^2 x \times \sec^2 x dx}{\tan^4 x + \tan^2 x + 1}$$

We know that $\sec^2 x = 1 + \tan^2 x$

Thus,

$$I = \int \frac{(1+\tan^2 x)\sec^2 x dx}{\tan^4 x + \tan^2 x + 1}$$

Now substitute $t = \tan x$; $dt = \sec^2 x dx$

Therefore,

$$I = \int \frac{(1+t^2)dt}{1+t^2+t^4}$$

Let us rewrite the integrand as

$$\frac{(1+t^2)}{1+t^2+t^4} = \frac{(1+t^2)}{(t^2-t+1)(t^2+t+1)}$$

Using partial fractions, we have

$$\frac{(1+t^2)}{1+t^2+t^4} = \frac{At+B}{t^2-t+1} + \frac{Ct+D}{t^2+t+1}$$

$$\Rightarrow \frac{(1+t^2)}{1+t^2+t^4} = \frac{(At+B)(t^2+t+1)+(Ct+D)(t^2-t+1)}{(t^2-t+1)(t^2+t+1)}$$

$$\Rightarrow \frac{(1+t^2)}{1+t^2+t^4}$$

$$= \frac{At^3 + At^2 + At + Bt^2 + Bt + B + Ct^3 - Ct^2 + Ct + Dt^2 - Dt + D}{(t^2-t+1)(t^2+t+1)}$$

$$\Rightarrow \frac{\left(1+t^2\right)}{1+t^2+t^4}$$
$$=\frac{t^3\left(A+C\right)+t^2\left(A+B-C+D\right)+t\left(A+B+C-D\right)+\left(B+D\right)}{\left(t^2-t+1\right)\left(t^2+t+1\right)}$$

So we have,

$A+C=0; A+B-C+D=1; A+B+C-D=0; B+D=1$

Solving the above equations, we have

$A=C=0 \text{ and } B=D=\frac{1}{2}$

$$I=\int\frac{\left(1+t^2\right)dt}{1+t^2+t^4}$$
$$=\int\left[\frac{1}{2\left(t^2-t+1\right)}+\frac{1}{2\left(t^2+t+1\right)}\right]dt$$
$$=\int\frac{dt}{2\left(t^2-t+1\right)}+\int\frac{dt}{2\left(t^2+t+1\right)}$$
$$=\frac{1}{2}\int\frac{dt}{t^2-t+1}+\frac{1}{2}\int\frac{dt}{t^2+t+1}$$
$$=I_1+I_2$$

where, $I_1=\frac{1}{2}\int\frac{dt}{t^2-t+1}$ and $I_2=\frac{1}{2}\int\frac{dt}{t^2+t+1}$

Consider I_1 :

$$I_1=\frac{1}{2}\int\frac{dt}{t^2-t+1}$$
$$=\frac{1}{2}\int\frac{dt}{t^2-t+\frac{1}{4}+1-\frac{1}{4}}$$
$$=\frac{1}{2}\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}$$
$$=\frac{1}{2}\times\frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)$$
$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2t-1}{\sqrt{3}}$$
$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2\tan x-1}{\sqrt{3}}$$

Similarly,

Consider I_2 :

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2+t+1}$$

$$= \frac{1}{2}\int \frac{dt}{t^2+t+\frac{1}{4}+1-\frac{1}{4}}$$

$$= \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{2\tan x+1}{\sqrt{3}}$$

Thus, $I = I_1 + I_2$

$$\Rightarrow I = \frac{1}{\sqrt{3}}\tan^{-1}\frac{2\tan x-1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2\tan x+1}{\sqrt{3}}$$

$$I = \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{2\tan x-1}{\sqrt{3}}+\tan^{-1}\frac{2\tan x+1}{\sqrt{3}}\right]+C$$

20. $\int_{-1} 7x-5\ dx$;

$a=-1,\ b=2;\ h=\dfrac{2+1}{n}\Rightarrow nh=3, f(x)=7x-5$

$$\underset{h\to 0}{\text{Lt}}\ h\left[f(-1)+f(-1+h)+f(-1+2h)+.....+f\left(-1+\overline{n-1}h\right)\right] \qquad \text{(i)}$$

$=f(-1)=-7-5=-12;\ f(-1+h)=7(-1+h)-5=7h-12$

$=f\left(-1+\overline{n-1}\,h\right)=7\left\{-1+\overline{n-1}\,h\right\}-5=7(n-1)h-12$

Substituting in (i)

$$\int_{-1}^{2}(7x-5)dx=\underset{h\to 0}{\text{Lt}}\ h\left[(-12)+(7h-12)+(14h-12)+.....+\{7(n-1)h-12\}\right]$$

$$=\underset{h\to 0}{\text{Lt}}\ h\left[7h\left(1+2+....+\overline{n-1}\right)-12n\right]=\underset{h\to 0}{\text{Lt}}\ h\left[7h\frac{(n-1)n}{2}-12n\right]$$

$$=\underset{h\to 0}{\text{Lt}}\left[\frac{7}{2}(nh)(nh-h)-12nh\right]=\underset{h\to 0}{\text{Lt}}\left[\frac{7}{2}(3)(3-h)-36\right]$$

$$=\frac{7}{2}\times 9-36=\frac{63}{2}-36=-\frac{9}{2}$$

21. The given D.E. can be written as

$$\frac{dy}{dx}=\frac{1}{\cos(x+y)}$$

let $x+y=v$. then,

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

D.E. becomes

$$\Rightarrow \frac{dv}{dx}-1=\frac{1}{\cos(v)}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1+\cos v}{\cos v}$$

$$\Rightarrow \int\frac{\cos v}{1+\cos v}dv=\int dx$$

$$\Rightarrow \int\frac{\cos v(1-\cos v)}{1-\cos^2 v}dv=\int dx$$

$$\Rightarrow \int\frac{\cos v-\cos^2 v}{\sin^2 v}dv=\int dx$$

$$\Rightarrow \int\cot v\,\mathrm{cosec}\,v\;dv-\int\cot^2 v\;dv=x+c$$

$$\Rightarrow \int\cot v\,\mathrm{cosec}\,v\;dv-\int\mathrm{cosec}^2 v\,dv+\int dv=x+c$$

$$\Rightarrow -\mathrm{cosec}\,v+\cot v+v=x+c$$

$$\Rightarrow -\mathrm{cosec}(x+y)+\cot(x+y)+(x+y)=x+c$$

$$\Rightarrow -\frac{1-\cos(x+y)}{\sin(x+y)}+y=c$$

$$\Rightarrow -\frac{2\sin^2\frac{x+y}{2}}{2\sin\frac{x+y}{2}\cos\frac{x+y}{2}}+y=c$$

$$\Rightarrow -\tan\left(\frac{x+y}{2}\right)+y=c$$

$$\Rightarrow -\operatorname{cosec}\ x+y\ +\cot\ x+y\ +\ x+y\ = x+c$$

$$\Rightarrow -\frac{1-\cos(x+y)}{\sin(x+y)}+y=c$$

$$\Rightarrow -\tan\left(\frac{x+y}{2}\right)+y=c$$

given $y=0$ when $x=0$.

$$\Rightarrow 0=c$$

so,

$$y=\tan\left(\frac{x+y}{2}\right)$$

as required.

OR

$$(x+y)^2\frac{dy}{dx}=a^2$$

let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

so,

$$v^2\left(\frac{dv}{dx}-1\right)=a^2$$

$$\Rightarrow v^2\frac{dv}{dx}-v^2=a^2$$

$$\Rightarrow v^2\frac{dv}{dx}=v^2+a^2$$

$$\Rightarrow \frac{v^2}{v^2+a}dv=dx$$

$$\Rightarrow \int\frac{v^2}{v^2+a}dv=\int dx$$

$$\Rightarrow \int\left(1-\frac{a^2}{v^2+a}\right)dv=\int dx$$

$$\Rightarrow v-a\tan^{-1}\frac{v}{a}=x+c$$

$$\Rightarrow (x+y)-a\tan^{-1}\left(\frac{x+y}{a}\right)=x+c$$

as required

22.

a) Given $\vec{\alpha}=3\hat{i}-\hat{j},\ \vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$

Since $\vec{\beta}_1 || \vec{\alpha}\ \therefore$ let $\vec{\beta}_1=\lambda\vec{\alpha}=3\lambda\hat{i}-\lambda\hat{j}$

Now

$\vec{\beta}_2=\vec{\beta}-\vec{\beta}_1=(2\hat{i}+\hat{j}-3\hat{k})-((3\lambda)\hat{i}-\lambda\hat{j})=(2-3\lambda)\hat{i}+(1+\lambda)\hat{j}-3\hat{k}$

since $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$

we get $\vec{\alpha}.\vec{\beta}_2=0$

$\Rightarrow 3(2-3\lambda)-(1+\lambda)=0$

$\Rightarrow \lambda=\frac{1}{2}$

Hence $\vec{\beta}_1=\frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}$ and $\vec{\beta}_2=\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$

b) Since each of the vectors is perpendicular to the sum of other two

$\Rightarrow \vec{a}.(\vec{b}+\vec{c})=0,\qquad \vec{b}.(\vec{c}+\vec{a})=0,\quad \vec{c}.(\vec{a}+\vec{b})=0$

Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$=\vec{a}.\vec{a}+\vec{a}.(\vec{b}+\vec{c})+\vec{b}.\vec{b}+\vec{b}.(\vec{c}+\vec{a})+\vec{c}.\vec{c}+\vec{c}.(\vec{a}+\vec{b})$

$=|\vec{a}|^2+0+|\vec{b}|^2+0+|\vec{c}|^2+0$

$=3^2+4^2+5^2=50$ Hence $|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$

23.

The equation of the given line is $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ ....(1)

Any point on the given line is $(3\lambda+2, 4\lambda-1, 2\lambda+2)$.

If this point lies on the given plane $x-y+z-5=0$, then

$3\lambda+2-(4\lambda-1)+2\lambda+2-5=0$

$\Rightarrow \lambda=0$

Putting $\lambda=0$ in $(3\lambda+2, 4\lambda-1, 2\lambda+2)$, we get the point of intersection of the given line and the plane is (2, -1, 2).

Let θ be the angle between the given line and the plane.

$\therefore \sin\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{(3\vec{i}+4\vec{j}+2\vec{k}).(\vec{i}-\vec{j}+\vec{k})}{\sqrt{3^2+4^2+2^2}\sqrt{1^2+1^2+1^2}}=\frac{3-4+2}{\sqrt{29}\sqrt{3}}=\frac{1}{\sqrt{87}}$

$\Rightarrow \theta=\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$

Thus, the angle between the given line and the given plane is $\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$

SECTION – D

24.

We have $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix}$ and $f(x) = x^3 - 6x^2 + 7x + 2$

$\therefore f(A) = A^3 - 6A^2 + 7A + 2I$

Now,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix}$$

$$A.A = A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 4 & 4 & 10 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^2.A = A^3 = \begin{bmatrix} 5 & 0 & 8 \\ 4 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 24 & 8 & 46 \\ 34 & 0 & 55 \end{bmatrix}$$

$$f(A) = A^3 - 6A^2 + 7A + 2I$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 24 & 8 & 46 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 4 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence, A is the root of the polynomials $f(x) = x^3 - 6x^2 + 7x + 2$.

OR

$$A=\begin{bmatrix}-1&-1&0\\0&-1&1\\2&3&4\end{bmatrix} \text{ and } B=\begin{bmatrix}1&2&3\\0&1&0\\1&1&0\end{bmatrix}$$

$$AB=\begin{bmatrix}-1&-1&0\\0&-1&1\\2&3&4\end{bmatrix}\begin{bmatrix}1&2&3\\0&1&0\\1&1&0\end{bmatrix}$$

$$AB=\begin{bmatrix}-1&-3&-3\\1&0&0\\6&11&6\end{bmatrix}$$

$$BA=\begin{bmatrix}1&2&3\\0&1&0\\1&1&0\end{bmatrix}\begin{bmatrix}-1&-1&0\\0&-1&1\\2&3&4\end{bmatrix}$$

$$BA=\begin{bmatrix}5&6&14\\0&-1&1\\-1&-2&1\end{bmatrix}$$

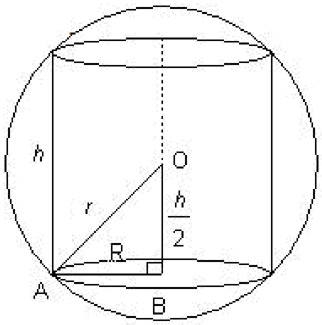
thus on comparing we get $AB \neq BA$

25. The given sphere is of radius r. Let *h* be the height and *R* be the radius of the cylinder inscribed in the sphere.

Volume of cylinder

$V=\pi R^2 h$ ...(1)

In right ΔOBA



$$AB^2 + OB^2 = OA^2$$

$$R^2 + \frac{h^2}{4} = r^2$$

$$\text{So, } R^2 = r^2 - \frac{h^2}{4}$$

Putting the value of R^2 in equation (1), we get

$$V = \pi\left(r^2 - \frac{h^2}{4}\right).h$$

$$V = \pi\left(r^2 h - \frac{h^3}{4}\right) \quad ...(3)$$

$$\therefore \quad \frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right) \quad ...(4)$$

For stationary point, $\frac{dV}{dh} = 0$

$$\pi\left(r^2 - \frac{3h^2}{4}\right) = 0$$

$$r^2 = \frac{3h^2}{4} \Rightarrow h^2 = \frac{4r^2}{3} \quad \Rightarrow h = \frac{2r}{\sqrt{3}}$$

Now $$\frac{d^2V}{dh^2} = \pi\left(-\frac{6}{4}h\right)$$

$\therefore$ $$\left[\frac{d^2V}{dh^2}\right]_{at\ h=\frac{2r}{\sqrt{3}}} = \pi\left(-\frac{3}{2}.\frac{2r}{\sqrt{3}}\right) < 0$$

$\therefore$ Volume is maximum at $h = \frac{2r}{\sqrt{3}}$

Maximum volume is

$$= \pi\left(r^2.\frac{2r}{\sqrt{3}} - \frac{1}{4}.\frac{8r^3}{3\sqrt{3}}\right)$$

$$= \pi\left(\frac{2r^3}{\sqrt{3}} - \frac{2r^3}{3\sqrt{3}}\right)$$

$$= \pi\left(\frac{6r^3 - 2r^3}{3\sqrt{3}}\right)$$

$$= \frac{4\pi r^3}{3\sqrt{3}} \text{ cu. unit}$$

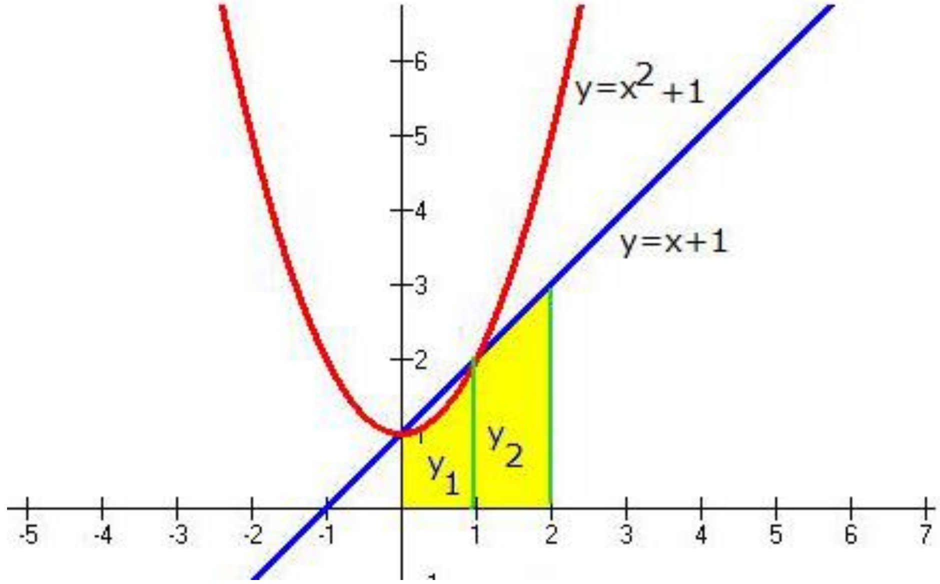
26. Points of intersection of $y = x^2 + 1$, $y = x + 1$

$x^2 + 1 = x + 1$

$\Rightarrow x(x - 1) = 0$

$\Rightarrow x = 0, 1$

So points of intersection are P(0, 1) and Q(1, 2). The graph is represented as



Required area is given by

$$A = \int_0^1 y_1\, dx + \int_1^2 y_2\, dx,$$

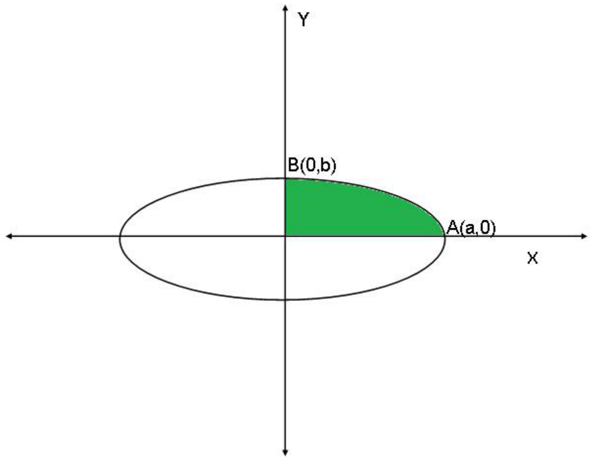
Where y_1 and y_2 represent the y co-ordinate of the parabola and straight line respectively.

$$\therefore A = \int_0^1 \left(x^2 + 1\right) dx + \int_1^2 (x + 1)\, dx$$

$$= \left(\frac{x^3}{3} + x\right)\Bigg]_0^1 + \left(\frac{x^2}{2} + x\right)_1^2$$

$$= \left[\left(\frac{1}{3} + 1\right) - 0\right] + \left[(2 + 2) - \left(\frac{1}{2} + 1\right)\right] = \frac{23}{6} \text{ sq. units}$$

OR



(i) between the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the x-axis between x = 0 to x = a

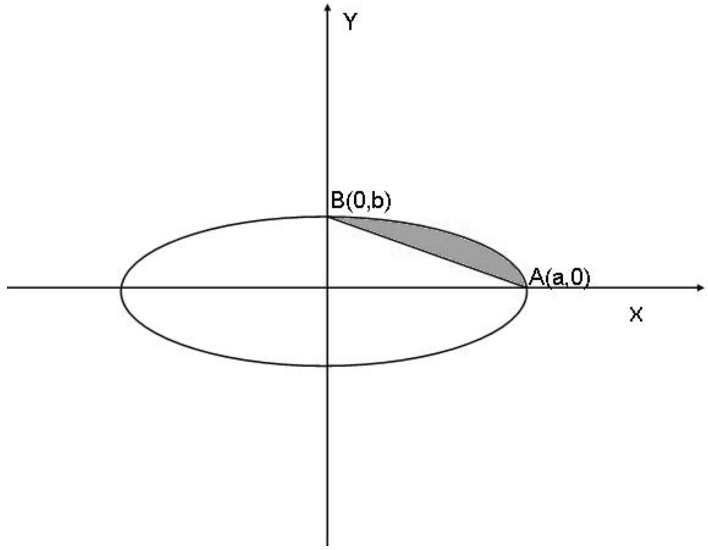
$$\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}dx$$

$$=\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{b}{2a}\left[\left(0+a^2\sin^{-1}(1)\right)-\left(0+a^2\sin^{-1}(0)\right)\right]$$

$$=\frac{b}{2a}\left[a^2\times\frac{\pi}{2}\right]$$

$$\therefore\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx=\frac{1}{4}\pi ab$$



(ii) Area of triangle AOB is in the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$

where OA = a and OB = b.

=the area enclosed between the chord AB and the arc AB of the ellipse .

= Area of Ellipse (In quadrant I)- Area of $\Delta AOB=\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx-\frac{1}{2}ab=\frac{1}{4}\pi ab-\frac{1}{2}ab$

$=\frac{(\pi-2)ab}{4}$

(iii) Ratio $=\frac{\frac{1}{4}\pi ab}{\frac{(\pi-2)}{4}ab}=\frac{\pi}{\pi-2}$

27. Let the equation of plane be ax + by + cz + d = 0 .... (1)

Since the plane passes through the point A (0, 0, 0) and B(3, -1, 2), we have

a × 0 + b × 0 + c × 0 + d = 0

$\Rightarrow$ d = 0 .... (2)

Similarly for point B (3, -1, 2), a × 3 + b × (-1) + c × 2 + d = 0

3a – b + 2c = 0 (Using, d = 0) ...(3)

Given equation of the line is $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$

We can also write the above equation as $\frac{x-4}{1}=\frac{y-(-3)}{-4}=\frac{x-(-1)}{7}$

The required plane is parallel to the above line.

Therefore, a × 1 + b × (-4) + c × 7= 0

$\Rightarrow$ a – 4b + 7c = 0 ... (4)

Cross multiplying equations (3) and (4), we obtain:

$$\frac{a}{(-1)\times7-(-4)\times2}=\frac{b}{2\times1-3\times7}=\frac{c}{3\times(-4)-1\times(-1)}$$

$$\Rightarrow\frac{a}{-7+8}=\frac{b}{2-21}=\frac{c}{-12+1}$$

$$\Rightarrow\frac{a}{1}=\frac{b}{-19}=\frac{c}{-11}=k$$

$$\Rightarrow a=k, b=-19k, c=-11k$$

Substituting the values of a, b and c in equation (1), we obtain the equation of plane as:

$$kx-19ky-11kz+d=0$$

$$\Rightarrow k(x-19y-11z)=0 \qquad \text{(From equation(2))}$$

$$\Rightarrow x-19y-11z=0$$

So, the equation of the required plane is x – 19y - 11z = 0

OR

We know that, equation of a line passing through x_1, y_1, z_1 with direction ratios a, b, c is given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

So, the required equation of a line passing through $(-1, 3, -2)$ is:

$$\frac{x+1}{a} = \frac{y-3}{b} = \frac{z+2}{c} ------(1)$$

Given that line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is perpendicular to line (1), so

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$a\ \ 1 + b\ \ 2 + c\ \ 3 = 0$$

$$a + 2b + 3c = 0 \qquad --- \ 2$$

And line $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$ is perpendicular to line 1, so

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$a\ \ -3 + b\ \ 2 + c\ \ 5 = 0$$

$$-3a + 2b + 5c = 0 \qquad --- \ 3$$

Solving equation 2 and 3 by cross multiplication,

$$\frac{a}{(2)(5)-(2)(3)} = \frac{b}{(-3)(3)-(1)(5)} = \frac{c}{(1)(2)-(2)(-3)}$$

$$\Rightarrow \frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6}$$

$$\Rightarrow \frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

$$\Rightarrow \frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = \lambda \text{ (Say)}$$

$$\Rightarrow a = 2\lambda,\ b = -7\lambda,\ c = 4\lambda$$

Putting the value of a, b, and c in (1) gives

$$\frac{x+1}{2\lambda} = \frac{y-3}{-7\lambda} = \frac{z+2}{4\lambda}$$

$$\Rightarrow \frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

28. Suppose x is the number of pieces of Model A and y is the number of pieces of Model B. Then

Total profit (in Rs.) = $8000x + 12000y$

Let Z = $8000x + 12000y$

Mathematical model for the given problem is as follows:

Maximise Z = $8000\,x + 12000\,y$ ... (1)

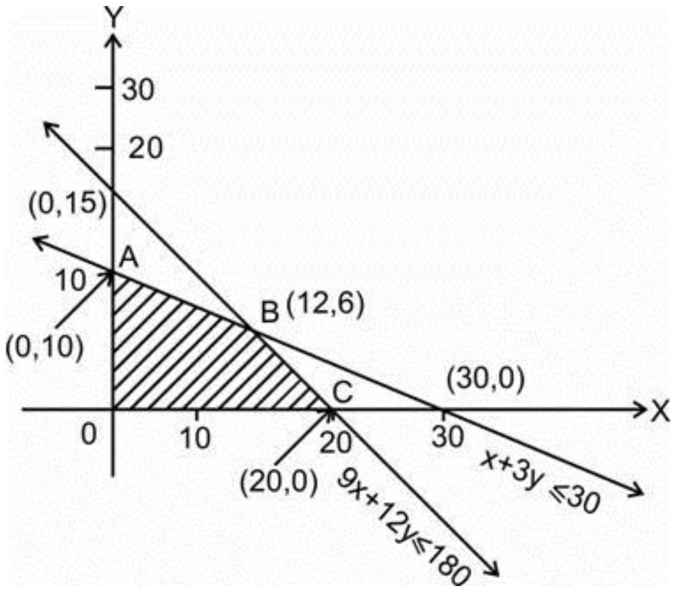
subject to the constraints,

$9x + 12y \le 180$ (Fabrication constraint) i.e. $3x + 4y \le 60$ .... (2)

$x + 3y \le 30$ (Finishing constraint) .... (3)

$x \ge 0, y \ge 0$ .... (4)

The feasible region (shaded) OABC determined by the linear inequalities (2) to (4) is shown below.



Corner Point	Z = $8000x$ + $12000y$
Corner Point	Z = $8000x$ + $12000y$
A(0, 10)	120000
B(12, 6)	168000← Maximum
C(20, 0)	16000

The company should produce 12 pieces of Model A and 6 pieces of Model B to realise maximum profit and maximum profit then will be Rs. 1,68,000.

29.

Let E_1 be the event that a red ball is transferred from bag A to bag B

Let E_2 be the event that a black ball is transferred from bag A to bag B

$\therefore E_1$ and E_2 are mutually exclusive and exhaustive.

$P(E_1) = 3/7$; $P(E_2) = 4/7$

Let E be the event that a red ball is drawn from bag B

$$P(E|E_1) = \frac{4+1}{(4+1)+5} = \frac{5}{10} = \frac{1}{2}$$

$$P(E|E_2) = \frac{3+1}{(5+1)+4} = \frac{4}{10} = \frac{2}{5}$$

$$\therefore \text{ Required probability } = P(E_2|E) = \frac{P(E|E_2)P(E_2)}{P(E|E_1)P(E_1)+P(E|E_2)P(E_2)}$$

$$= \frac{\frac{4}{10}\times\frac{4}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{4}{10}\times\frac{4}{7}} = \frac{\frac{16}{70}}{\frac{3}{14}+\frac{16}{70}} = \frac{\frac{16}{70}}{\frac{31}{70}} = \frac{16}{31}$$

$$\therefore \text{ Required probability } = P(E_1|E) = \frac{P(E|E_1)P(E_1)}{P(E|E_1)P(E_1)+P(E|E_2)P(E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{3}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{4}{10}\times\frac{4}{7}} = \frac{\frac{3}{14}}{\frac{3}{14}+\frac{16}{70}} = \frac{\frac{3}{14}}{\frac{31}{70}} = \frac{15}{31}$$